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LETTER TO THE EDITOR

Path-integral approach to the critical behaviour of two-dimensional systems

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Abstract. A path-integral approach to the critical behaviour of two-dimensional systems is presented. The method, based on a decoupling change of path-integral variables, enables one to compute critical exponents in a simple and systematic way. Thus, it provides a new tool to be used in the field-theoretical description of certain statistical mechanics models.

Some years ago a path-integral approach to bosonisation was developed which has been shown to be very useful in the study of two-dimensional (2D) quantum field theories with Abelian and non-Abelian symmetries [1–4]. Fujikawa's [5] observation on the non-invariance of the path-integral measure under γ_5 (chiral) transformations plays a crucial role in this method. The purpose of this letter is to show how this technique can be extended for use in the context of 2D statistical mechanics systems, allowing us to obtain critical exponents in a simple way. As examples, in order to illustrate the approach, we shall consider the continuous versions of the Ising and Baxter [6] models. In particular we will rederive the critical behaviour of the Ising spin–spin correlator and the energy density and crossover operators in the Baxter model.

We start from the square of the Ising correlation function in the critical regime, as given by Zuber and Itzykson [7],

$$\langle \sigma_0 \sigma_R \rangle^2 = \left\langle \exp\left(\pi \int_0^R \mathrm{d}x_1\, J_0(x_1) \right) \right\rangle \tag{1}$$

where $J_\mu = \bar\psi \gamma_\mu \psi$ is the Dirac fermion current and $\langle\ \rangle$ in the right-hand side is the vacuum expectation value in a model of free massless fermion fields. Note that the squaring allows us to work with Dirac fermions instead of the Majorana fermions which arise in the original transfer matrix formalism [8]. We work in Euclidean 2D spacetime with matrices chosen in the form

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \qquad \gamma_5 = \mathrm{i}\gamma_0\gamma_1 .$$

In [7] the R dependence of the correlator was obtained by using Mandelstam's [9] bosonisation prescriptions. In this letter we shall follow a different route based on a decoupling change of variables in the fermionic path integral measure. To this end, let us rewrite the line integral in (1) as

$$\int_0^R J_0(x_1)\, \mathrm{d}x_1 = \int \mathrm{d}^2x\, \bar\psi \not{A} \psi$$

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where we have introduced the auxiliary vector field A_μ with components

$$A_0 = \delta(x_0)\theta(x_1)\theta(R-x_1) \qquad A_1 = 0. \tag{2}$$

A_μ can be written as

$$A_\mu = \varepsilon_{\mu\nu}\partial_\nu\phi + \partial_\mu\eta \tag{3}$$

where according to (2), the classical fields ϕ and η must satisfy

$$\begin{aligned} &-\partial_0\phi + \partial_1\eta = 0 \\ &\partial_1\phi + \partial_0\eta = \delta(x_0)\theta(x_1)\theta(R-x_1). \end{aligned} \tag{4}$$

In this new language (1) can be expressed in terms of fermionic determinants:

$$\langle\sigma_0\sigma_R\rangle^2 = \frac{\int \mathscr{D}\bar\psi\,\mathscr{D}\psi\,\exp(-\int \mathrm{d}^2x\,\bar\psi(\mathrm{i}\not\partial + g\not A)\psi)}{\int \mathscr{D}\bar\psi\,\mathscr{D}\psi\,\exp(-\int \mathrm{d}^2x\bar\psi\mathrm{i}\not\partial\psi)} = \frac{\det(\mathrm{i}\not\partial + g\not A)}{\det(\mathrm{i}\not\partial)}. \tag{5}$$

The constant g ($g = \pi$ in the present case) has been introduced for later convenience. At this point we perform the following change of path-integral fermionic variables:

$$\begin{aligned} \psi &= \exp[-g(\gamma_5\phi - \mathrm{i}\eta)]\chi \\ \bar\psi &= \bar\chi\,\exp[-g(\gamma_5\phi + \mathrm{i}\eta)] \end{aligned} \tag{6}$$

which has been chosen so as to cancel the coupling between scalars and fermions in the kinetic term of the effective Lagrangian

$$\begin{aligned} \mathscr{L}_{\mathrm{eff}} = \bar\chi\,&\exp[-g(\gamma_5\phi + \mathrm{i}\eta)]\mathrm{i}\not\partial\,\exp[-g(\gamma_5\phi - \mathrm{i}\eta)]\chi \\ &+ g\bar\chi\,\exp[-g(\gamma_5\phi + \mathrm{i}\eta)]\not A\,\exp[-g(\gamma_5\phi - \mathrm{i}\eta)]\chi \equiv \bar\chi\mathrm{i}\not\partial\chi. \end{aligned}$$

Taking into account the fermionic Jacobian $J_{\mathrm{F}}(\mathscr{D}\bar\psi\,\mathscr{D}\psi = J_{\mathrm{F}}\,\mathscr{D}\bar\chi\,\mathscr{D}\chi)$ one obtains $\det(\mathrm{i}\not\partial + g\not A) = J_{\mathrm{F}}\det(\mathrm{i}\not\partial)$ and the squared correlator becomes

$$\langle\sigma_0\sigma_R\rangle^2 = J_{\mathrm{F}}. \tag{7}$$

The fermion Jacobian is not trivial due to the non-invariance of the measure under chiral changes. It can be computed following Fujikawa's procedure [5]. Its detailed derivation has been given several times in the literature [1-4]; here we just write down the result:

$$\ln J_{\mathrm{F}} = -\frac{g^2}{2\pi}\int \mathrm{d}^2x\,\phi(x)\varepsilon_{\mu\nu}\partial_\mu A_\nu. \tag{8}$$

In our particular case ($g = \pi$), using (2) and (3) we obtain

$$\ln\langle\sigma_0\sigma_R\rangle = \tfrac{1}{2}\ln J_{\mathrm{F}} = -\tfrac{1}{4}\pi[\phi(R) - \phi(0)]. \tag{9}$$

From (4) it follows that

$$\Box\phi(x) = \delta(x_0)\frac{\mathrm{d}}{\mathrm{d}x_1}[\theta(x_1)\theta(R-x_1)]$$

which can be easily solved by considering the convolution

$$\phi(x) = \int \mathrm{d}^2x'\,G(x,x')\delta(x_0')\frac{\mathrm{d}}{\mathrm{d}x_1'}[\theta(x_1')\theta(R-x_1')]$$

with $\Box G(x, x') = \delta^2(x-x')$ and $G(x, x') = (1/4\pi)\ln(|x-x'|^2+a^2)$, where a is an ultraviolet cutoff provided by the lattice spacing. One then gets

$$\phi(x) = -\frac{1}{4\pi}\ln\left(\frac{x_0^2+(x_1-R)^2+a^2}{x_0^2+x_1^2+a^2}\right).$$

If this expression is substituted into (9) then, for large R, we readily obtain

$$\langle\sigma_0\sigma_R\rangle \simeq (a/R)^{1/4} \tag{10}$$

which is the well known critical behaviour of the Ising spin–spin correlator.

Let us now apply our formulation to the 2D Baxter model [6]. This system can be considered as two 2D Ising models coupled by four-spin interactions. Its scaling limit near the critical point is described by the Thirring model [10] with Lagrangian

$$\mathscr{L} = \bar\psi \mathrm{i}\not\partial\psi - \tfrac{1}{2}g^2(\bar\psi\gamma_\mu\psi)^2$$

where the parameter g measures the strength of the four-spin coupling (the case $g=0$ corresponds to two independent Ising models). We shall evaluate the critical exponents of the energy density (Δ_ε) and crossover (Δ_{Cr}) [11] operators:

$$\varepsilon(x) = \bar\psi\psi(x) = \bar\psi_1\psi_1(x) + \bar\psi_2\psi_2(x) \tag{11}$$

$$\mathrm{Cr}(x) = \bar\psi_2\bar\psi_1 - \psi_1\psi_2 \tag{12}$$

with

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \qquad \bar\psi = (\bar\psi_1\bar\psi_2). \tag{13}$$

First we consider the energy-density correlation function

$$\langle\varepsilon(x)\varepsilon(y)\rangle = N\int \mathscr{D}\bar\psi\,\mathscr{D}\psi\,\exp(-\textstyle\int \mathrm{d}^2x\,\mathscr{L})\varepsilon(x)\varepsilon(y) \tag{14}$$

where N is a normalisation factor. In order to eliminate the quartic interaction in $\mathscr{L}$ we use the identity [3]

$$\exp\left(\tfrac{1}{2}g^2\int \mathrm{d}^2x(\bar\psi\gamma_\mu\psi)^2\right) = \int \mathscr{D}A_\mu\,\exp\left(-\int \mathrm{d}^2x(\tfrac{1}{2}A_\mu^2 + g\bar\psi\not{A}\psi)\right). \tag{15}$$

Here A_μ is an auxiliary vector field which (in two dimensions) can be written in the form

$$A_\mu = \varepsilon_{\mu\nu}\partial_\nu\phi + \partial_\mu\eta \tag{16}$$

exactly as we did in the previous calculation of the Ising correlator. However, we want to stress that in the present case ϕ and η are quantum fields whose dynamics plays a crucial role in the following computation.

Using (14)–(16) we find

$$\langle\varepsilon(x)\varepsilon(y)\rangle = N\int \mathscr{D}\bar\psi\,\mathscr{D}\psi\,\mathscr{D}\phi\,\mathscr{D}\eta\,\exp\left(-\int \mathrm{d}^2x\,\mathscr{L}_{\mathrm{eff}}\right)\varepsilon(x)\varepsilon(y) \tag{17}$$

where

$$\mathscr{L}_{\mathrm{eff}} = \bar\psi[\mathrm{i}\not\partial - g\gamma_\mu(\varepsilon_{\mu\nu}\partial_\nu\phi + \partial_\mu\eta)]\psi + \tfrac{1}{2}[(\partial_\mu\phi)^2 + (\partial_\mu\eta)^2].$$

Again, we perform the decoupling change (6), which allows us to write the correlator (17) in terms of new variables provided the corresponding Jacobians are taken into account:

$$\mathcal{D}\bar{\psi}\,\mathcal{D}\psi = J_{\mathrm{F}}\,\mathcal{D}\bar{\chi}\,\mathcal{D}\chi \tag{18}$$

$$\mathcal{D}A_\mu = J_{\mathrm{A}}\,\mathcal{D}\phi\,\mathcal{D}\eta. \tag{19}$$

The fermion Jacobian in (18) is formally given, as before, by equation (8) where now, of course, ϕ and η are scalar fields subjected to quantum fluctuations. Concerning the change (19), it trivially yields $J_{\mathrm{A}} = \det\Box$, which can be absorbed in the normalisation constant. We then have

$$\langle\varepsilon(x)\varepsilon(y)\rangle = N\int \mathcal{D}\bar{\chi}\,\mathcal{D}\chi\,\mathcal{D}\phi\,\mathcal{D}\eta\;\tilde{\varepsilon}(x)\tilde{\varepsilon}(y)$$
$$\times\exp\left(-\int \mathrm{d}^2x[\bar{\chi}\mathrm{i}\not{\partial}\chi + \tfrac{1}{2}(1+g^2/\pi)(\partial_\mu\phi)^2 + \tfrac{1}{2}(\partial_\mu\eta)^2]\right) \tag{20}$$

where $\tilde{\varepsilon}(x)$ is the transformed operator

$$\tilde{\varepsilon}(x) = \bar{\chi}(x)\exp(2g\gamma_5\phi(x))\chi(x). \tag{21}$$

From now on, the procedure is quite similar to that employed in the path-integral bosonisation of massive fermions [4]. First of all we observe that $\tilde{\varepsilon}$ is independent of η and, therefore, the η-dependent piece of the action in (20) can be absorbed in N. We are then left with the evaluation of $\langle\tilde{\varepsilon}(x)\tilde{\varepsilon}(y)\rangle$ in a theory of free massless fermions and scalars with propagators

$$G_{\mathrm{F}}(x) = \frac{\mathrm{i}}{2\pi}\frac{\gamma_\mu x_\mu}{x^2} \tag{22}$$

$$\Delta_{\mathrm{F}}(x) = \frac{\lambda^2}{2\pi}K_0(\mu x) \tag{23}$$

where $\lambda^2 = 1/(1+g^2/\pi)$ and $K_0(\mu x)$ is the zeroth-order modified Bessel function ($K_0(z) \to_{z\to\infty} 0$, $K_0(z) \mathop{—}_{z\to 0} -\ln(z) + \text{constant}$). The small mass μ has been introduced to avoid the infrared problem which is typical of 2D models. Of course, we shall take $\mu \to 0$ at the end of our computation.

In order to compute (20) one just separates the boson factor from the free fermionic part by writing

$$\bar{\chi}\exp(2g\gamma_5\phi)\chi = \exp(2g\phi)\bar{\chi}\tfrac{1}{2}(1+\gamma_5)\chi + \exp(-2g\phi)\bar{\chi}\tfrac{1}{2}(1-\gamma_5)\chi$$

and uses the identity [4, 12]

$$\left\langle \exp\left(\mathrm{i}\sum_j \beta_j\phi(x_j)\right)\right\rangle_{\mathrm{bos}} = \left(\frac{\mu}{\rho}\right)^{(\lambda^2/4\pi)(\Sigma_i\beta_i)^2}\left(\frac{\rho}{\Lambda}\right)^{(\lambda^2/4\pi)\Sigma_i\beta_i^2}\prod_{i>j}(\rho c|x_i - x_j|)^{(\lambda^2/2\pi)\beta_i\beta_j} \tag{24}$$

where Λ is a large mass introduced to cut off the free boson model and ρ is an arbitrary normal-ordering mass [12]. Note that if $\Sigma_i\beta_i \neq 0$, then (24) vanishes in the limit $\mu\to 0$. Therefore we shall restrict our analysis to the 'neutral' case $\Sigma_i\beta_i = 0$ (in the context of the Coulomb gas model this is known as the 'neutrality condition'). Thus we obtain

$$\langle\varepsilon(x)\varepsilon(y)\rangle = N\langle\exp[2g(\phi(x)-\phi(y))]\rangle_{\mathrm{bosonic}}$$
$$\times\langle\bar{\chi}\tfrac{1}{2}(1+\gamma_5)\chi](x)[\bar{\chi}\tfrac{1}{2}(1-\gamma_5)\chi](y)\rangle_{\mathrm{fermionic}}. \tag{25}$$

Using (13) and (22), the fermionic part is readily computed by writing

$$\bar{\chi}\tfrac{1}{2}(1+\gamma_5)\chi=\bar{\chi}_1\chi_1 \qquad \bar{\chi}\tfrac{1}{2}(1-\gamma_5)\chi=\bar{\chi}_2\chi_2$$

and then by employing (24) for the bosonic part to obtain

$$\langle\varepsilon(x)\varepsilon(y)\rangle\simeq|x-y|^{-2/(1+g^2/\pi)}. \tag{26}$$

Therefore, the g-dependent critical exponent for the energy density is

$$\Delta_\varepsilon=\frac{1}{1+g^2/\pi}. \tag{27}$$

The evaluation of the crossover index can be performed following the same lines. The only difference is that, in this case, the transformed operator $\tilde{\mathrm{C}}\mathrm{r}(x)$ depends on η instead of ϕ. The corresponding correlation function behaves like

$$\begin{aligned}\langle \mathrm{Cr}(x)\mathrm{Cr}(y)\rangle &\simeq\langle\exp[2\mathrm{i}g(\eta(x)-\eta(y))]\rangle_{\text{bosonic}}\langle(\bar{\chi}_1\bar{\chi}_2)(x)(\chi_1\chi_2)(y)\rangle_{\text{fermionic}}\\ &\simeq|x-y|^{-2(1+g^2/\pi)}\end{aligned} \tag{28}$$

which yields

$$\Delta_{\mathrm{Cr}}=1+g^2/\pi=1/\Delta_\varepsilon. \tag{29}$$

This relation between both exponents had been predicted by several authors [13, 14] and was first derived by Drugowich de Felicio and Koberle [15] in the operator framework.

In conclusion, we have presented a path-integral approach, previously developed in the context of 2D quantum field theories, which has enabled us to obtain critical properties of 2D statistical mechanics systems in a systematic and straightforward way. In particular we were able to give a new derivation of the well known critical behaviour of the Ising spin-correlator. On the other hand, we computed the critical exponents of the energy density and crossover operators in the Baxter model, providing a rigorous proof of the relation $\Delta_{\mathrm{Cr}}=1/\Delta_\varepsilon$ in the path-integral framework. The analysis of models away from the critical point can be also envisaged in this approach following the lines discussed in [4]. Work on this aspect is in progress and will be reported elsewhere.

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References

[1] Roskies R and Schaposnik F A 1981 *Phys. Rev.* D **23** 558
[2] Gamboa Saraví R E Schaposnik F A and Solomin J E 1981 *Nucl. Phys.* B **185** 239
[3] Furuya K, Gamboa Saraví R E and Schaposnik F A 1982 *Nucl. Phys.* B **208** 159
[4] Naón C M 1985 *Phys. Rev.* D **31** 2035
[5] Fujikawa K 1979 *Phys. Rev. Lett.* **42** 1195; 1980 *Phys. Rev.* D **21** 2848
[6] Baxter R 1971 *Phys. Rev. Lett.* **26** 832
[7] Zuber J B and Itzykson C 1977 *Phys. Rev.* D **15** 2875
[8] Schultz T D, Mattis D C and Lieb E H 1964 *Rev. Mod. Phys.* **36** 856
[9] Mandelstam S 1975 *Phys. Rev.* D **11** 3026
[10] Luther A and Peschel I 1975 *Phys. Rev.* B **12** 3908
[11] Kadanoff L P 1979 *Ann. Phys., NY* **120** 39
[12] Coleman S 1975 *Phys. Rev.* D **11** 2088
[13] Enting I 1975 *J. Phys. A: Math. Gen.* **8** L35
[14] Kadanoff L P 1977 *Phys. Rev. Lett.* **39** 903
[15] Drugowich de Felicio J R and Koberle R 1982 *Phys. Rev.* B **25** 511